

EELE 477
Digital Signal Processing

10a

z-Transforms

z-Transform Definition

- For a finite length signal $x[n]$, $n=0\dots N$, the z-transform is defined:

$$\begin{aligned} X(z) &= \sum_{k=0}^N x[k] z^{-k} \\ &= x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \dots + x[N]z^{-N} \end{aligned}$$

- z is simply a variable representing any *complex number*, that is, z is a *complex variable*
- $X(z)$ is customarily expressed as a ratio of polynomials in z^{-k}

n -domain and z -domain

- Given $x[n]$, we can use the formula to “take the z -transform” and get $X(z)$.
- Given $X(z)$, we can “take the inverse z -transform” and get $x[n]$.
- Examples:

$$x[n] = \delta[n - n_0] \Leftrightarrow X(z) = z^{-n_0}$$

$$x[n] = 3\delta[n] - 6\delta[n - 1] + 2\delta[n - 2] + 7\delta[n - 3]$$

$$\Leftrightarrow X(z) = 3 - 6z^{-1} + 2z^{-2} + 7z^{-3}$$

z-transforms and Linear Systems

- Consider FIR filter (convolve h and x):

$$y[n] = \sum_{k=0}^M h[k] x[n-k] = \sum_{k=0}^M b_k x[n-k]$$

- We previously applied $x[n]=e^{j\omega n}$, all n , to define the *frequency response*.
- Now apply $x[n]=z^n$:

$$y[n] = \sum_{k=0}^M b_k z^{(n-k)} = \sum_{k=0}^M b_k z^n z^{-k} = \underbrace{\left(\sum_{k=0}^M b_k z^{-k} \right)}_{H(z)} \underbrace{z^n}_{x[n]}$$

System Function

- Note the last result: $H(z) = \sum_{k=0}^M b_k z^{-k} = \sum_{k=0}^M h[k] z^{-k}$

The system function $H(z)$ is the z -transform of the unit sample response.

- System function (product) is related to convolution, since if x is z^n

$$\begin{aligned} y[n] &= H(z) z^n \\ &= h[n] * z^n \end{aligned}$$

z-transform properties

- Linearity:

$$Z\{ax_1[n]+bx_2[n]\}=aZ\{x_1[n]\}+bZ\{x_2[n]\}$$

- Delay: multiply by z^{-1} corresponds to delay of one sample; multiply by z^{-n_0} corresponds to delay of n_0 .

- General infinite length z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Note: need to verify that the sum converges!!

Z-transform and convolution

- Take the z-transform of a convolution expression:

$$y[n] = x[n] * h[n] = \sum_{k=0}^M h[k]x[n-k]$$

$$Y(z) = Z\{h[0]x[n] + h[1]x[n-1] + \dots + h[M]x[n-M]\}$$

$$= \sum_{k=0}^M h[k](z^{-k} X(z)) = \left(\sum_{k=0}^M h[k]z^{-k} \right) X(z)$$

$$= H(z)X(z)$$

Sequence domain convolution \Leftrightarrow z-domain product

z-transform: polynomial in z^{-1}

- Consider an FIR function:

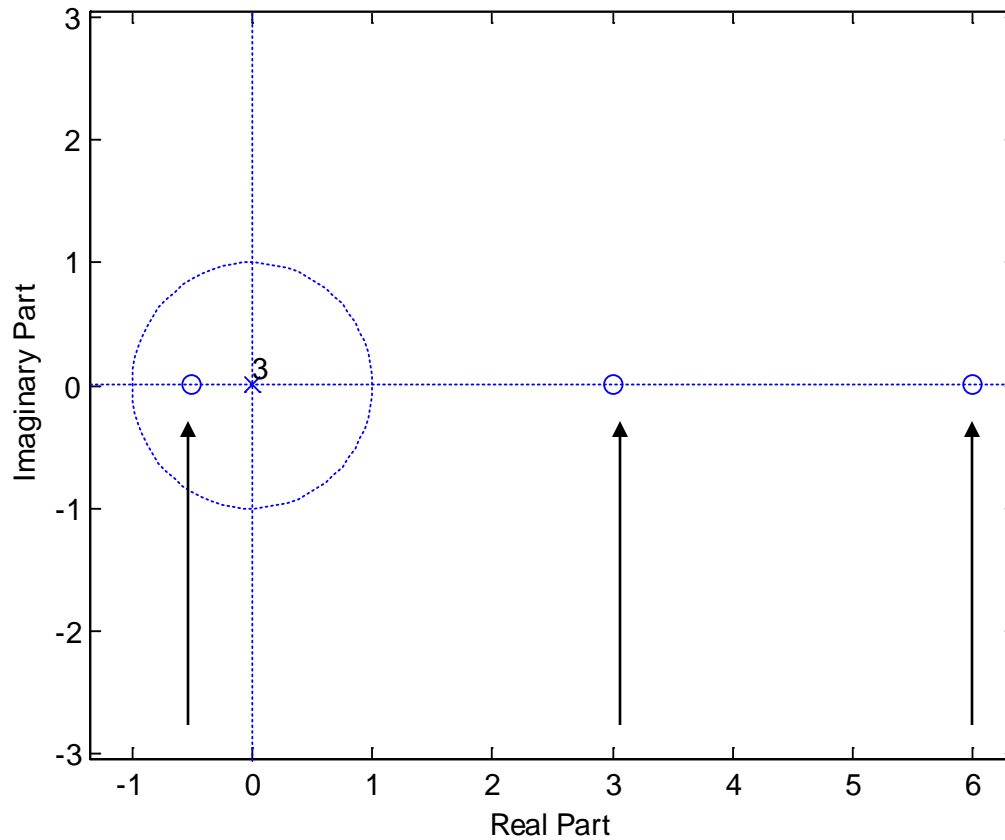
$$y[n] = x[n] - 8.5x[n-1] + 13.5x[n-2] + 9x[n-3]$$

$$Y(z) = X(z) - 8.5z^{-1}X(z) + 13.5z^{-2}X(z) + 9z^{-3}X(z)$$

$$\begin{aligned} H(z) &= Y(z) / X(z) = 1 - 8.5z^{-1} + 13.5z^{-2} + 9z^{-3} \\ &= (1 + 0.5z^{-1})(1 - 3z^{-1})(1 - 6z^{-1}) \end{aligned}$$

- Zeros of $H(z)$: $-.5, 3, 6$; 3 poles @ 0

Graphical Depiction of $H(z)$



Another $h[n] \leftrightarrow H(z)$ example

- Consider an FIR system:

$$h[n] = \delta[n] - 2\delta[n-1] + 5\delta[n-2] + 5\delta[n-3] - 2\delta[n-4] + \delta[n-5]$$

$$H(z) = 1 - 2z^{-1} + 5z^{-2} + 5z^{-3} - 2z^{-4} + z^{-5}$$

$$= (1 - 2.618z^{-1} + 6.8541z^{-2})(1 - 0.382z^{-1} + 0.1459z^{-2})(1 + z^{-1})$$

- System zeros:

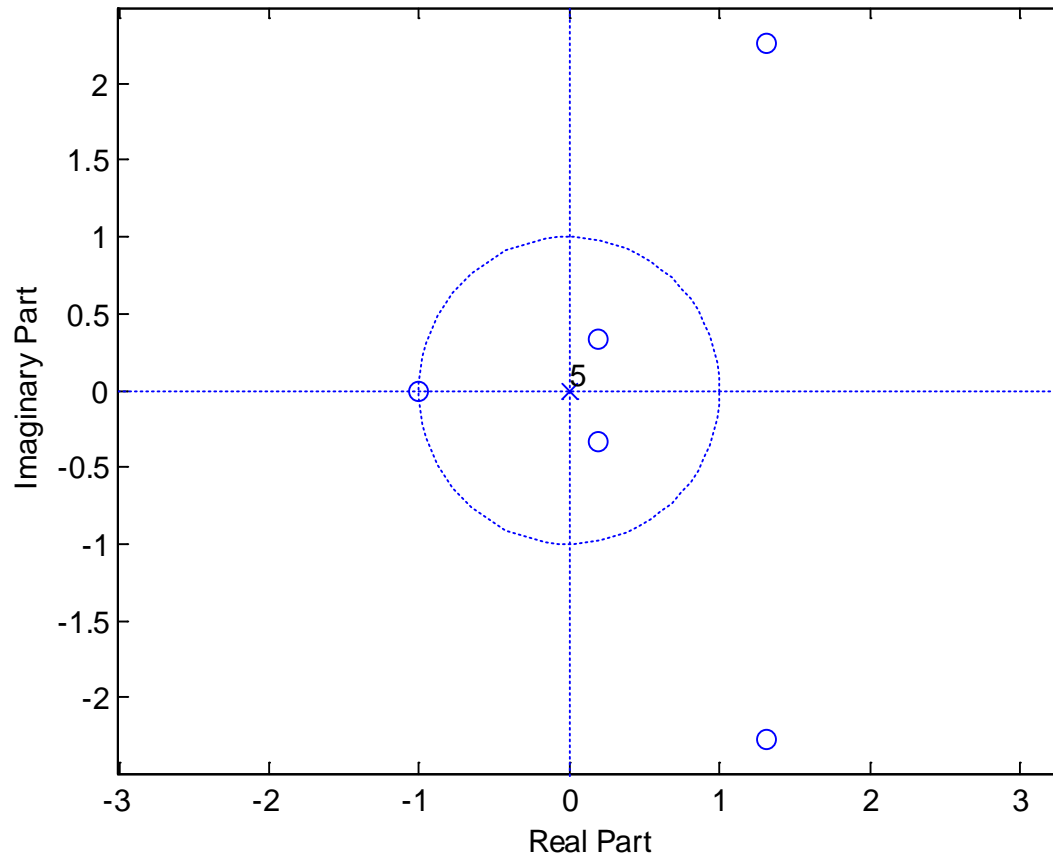
$$1.3090 + j2.2673, \quad 1.3090 - j2.2673$$

$$-1.0000$$

$$0.1910 + j0.3308, \quad 0.1910 - j0.3308$$

Z-plane Sketch

Complex conjugate zeros \Rightarrow real filter coefficients



Example z-transform product

- **Ex:** $x[n] = \delta[n - 1] - 5\delta[n - 2] + 8\delta[n - 8] - 4\delta[n - 9]$
 $h[n] = \delta[n] + \delta[n - 1] - 0.3\delta[n - 2]$
- **Compute using z-transform product:**

$$X(z) = z^{-1} - 5z^{-2} + 8z^{-8} - 4z^{-9}$$

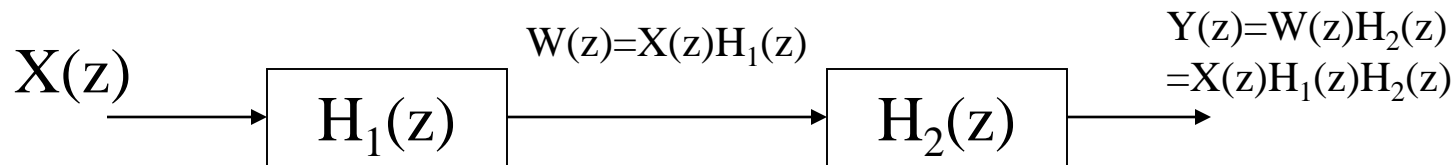
$$H(z) = 1 + z^{-1} - 0.3z^{-2}$$

$$Y(z) = X(z)H(z)$$

$$= z^{-1} - 4z^{-2} - 5.3z^{-3} + 1.5z^{-4} + 8z^{-8} + 4z^{-9} - 6.4z^{-10} + 1.2z^{-11}$$

Cascade of Systems

- Note cascade in terms of system functions:



- System function of cascade is the *product* of the system functions